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EXPONENTIAL DRIVING FUNCTION FOR THE LÖWNER EQUATION

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Abstract. We consider the chordal Löwner differential equation with the model driving function $\sqrt[3]{t}$. Holomorphic and singular solutions are represented by their series. It is shown that a disposition of values of different singular and branching solutions is monotonic, and solutions to the Löwner equation map slit domains onto the upper half-plane. The slit is a C^1-curve. We give an asymptotic estimate for the ratio of harmonic measures of the two slit sides.

1. Introduction

The Löwner differential equation introduced by K. Löwner [11] served a source to study properties of univalent functions on the unit disk. Nowadays it is of growing interest in many areas, see, e.g., [12]. The Löwner equation for the upper half-plane $\mathbb{H}$ appeared later (see, e.g., [1]) and became popular during the last decades. Define a function $w = f(z,t)$, $z \in \mathbb{H}$, $t \geq 0$,

$$f(z,t) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \tag{1}$$

which maps $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ and solves the *chordal* Löwner ordinary differential equation

$$\frac{df(z,t)}{dt} = \frac{2}{f(z,t) - \lambda(t)}, \quad f(z,0) = z, \quad z \in \mathbb{H}, \tag{2}$$

where the driving function $\lambda(t)$ is continuous and real-valued.

The conformal maps $f(z,t)$ are continuously extended onto $z \in \mathbb{R}$ minus the closure of K_t and the extended map also satisfies equation (2). Following [10], we pay attention to an old problem to determine, in terms of λ, when K_t is a Jordan arc, $K_t = \gamma(t)$, $t \geq 0$, emanating from the real axis $\mathbb{R}$. In this case $f(z,t)$ are continuously extended onto the two sides of $\gamma(t)$,

$$\lambda(t) = f(\gamma(t), t), \quad \gamma(t) = f^{-1}(\lambda(t), t). \tag{3}$$

Points $\gamma(t)$ are treated as prime ends which are different for the two sides of the arc. Note that Kufarev [9] proposed a counterexample of the non-slit mapping for the *radial* Löwner equation in the disk. For the chordal Löwner equation, Kufarev's example corresponds to $\lambda(t) = 3\sqrt{2}\sqrt{1-t}$, see [8], [10] for details.

Equation (2) admits integrating in quadratures for partial cases of $\lambda(t)$ studied in [8], [15]. The integrability cases of (2) are invariant under linear and scaling

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transformations of $\lambda(t)$, see, e.g., [10]. Therefore, assume without loss of generality that $\lambda(0) = 0$ and, equivalently, $\gamma(0) = 0$.

The picture of singularity lines for driving functions $\lambda(t)$ belonging to the Lipschitz class Lip(1/2) with the exponent 1/2 is well studied, see, e.g., [10] and references therein.

This article is aimed to show that in the case of the cubic root driving function $\lambda(t) = \sqrt[3]{t}$ in (2), that is,

$$\frac{df(z,t)}{dt} = \frac{2}{f(z,t) - \sqrt[3]{t}}, \quad f(z,0) = z, \quad \text{Im } z \geq 0, \tag{4}$$

the solution $w = f(z,t)$ is a slit mapping for $t > 0$ small enough, i.e., $K_t = \gamma(t)$, $0 < t < T$.

The driving function $\lambda(t) = \sqrt[3]{t}$ is chosen as a typical function of the Lipschitz class Lip(1/3). We do not try to cover the most general case but hope that the model driving function serves a demonstration for a wider class. By the way, the case when the trace γ is a circular arc meeting the real axis tangentially is studied in [14]. The explicit solution for the inverse function gave a driving term of the form $\lambda(t) = Ct^{\frac{1}{3}} + \dots$ which corresponds to the above driving function asymptotically.

The main result of the article is contained in the following theorem which shows that $f(z,t)$ is a mapping from a slit domain $D(t) = \mathbb{H} \setminus \gamma(t)$.

Theorem 1. *Let $f(z,t)$ be a solution to the Löwner equation (4). Then $f(\cdot,t)$ maps $D(t) = \mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$ for $t > 0$ small enough where $\gamma(t)$ is a C^1-curve, except probably for the point $\gamma(0) = 0$.*

Preliminary results of Section 2 in the article concern the theory of differential equations and preparations for the main proof.

Theorem 1 together with helpful lemmas are proved in Section 3.

Section 4 is devoted to estimates for harmonic measures of the two sides of the slit generated by the Löwner equation (4). Theorem 2 in this Section gives the asymptotic relation for the ratio of these harmonic measures as $t \to 0$.

In Section 5 we consider holomorphic solutions to (4) represented by power series and propose asymptotic expansions for the radius of convergence of the series.

2. Preliminary statements

Change variables $t \to \tau^3$, $g(z,\tau) := f(z,\tau^3)$, and reduce equation (4) to

$$\frac{dg(z,\tau)}{d\tau} = \frac{6\tau^2}{g(z,\tau) - \tau}, \quad g(z,0) = z, \quad \text{Im } z \geq 0. \tag{5}$$

Note that differential equations

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

with holomorphic functions $P(x,y)$ and $Q(x,y)$ are well known both for complex and real variables, especially in the case of polynomials P and Q, see, e.g., [2], [3], [4], [13], [16], [17].

If $z \neq 0$, then $g(z,0) \neq 0$, and there exists a *regular* solution $g(z,\tau)$ to (5) holomorphic in τ for $|\tau|$ small enough which is unique for every $z \neq 0$. We are interested mostly in studying *singular* solutions to (5), i.e., those which do not satisfy the uniqueness conditions for equation (5). Every point $(g(z_0,\tau_0),\tau_0)$ such that $g(z_0,\tau_0) = \tau_0$ is a *singular point* for equation (5). If $\tau_0 \neq 0$, then $(g(z_0,\tau_0),\tau_0)$ is an *algebraic solution critical point*, and corresponding singular solutions to (5) through this point are expanded in series in terms $(\tau - \tau_0)^{1/m}$, $m \in \mathbb{N}$. So these singular solutions are different branches of the same analytic function, see [17, Chap.9, §1].

The point $(g(z_0,\tau_0),\tau_0) = (0,0)$ is the only *singular point of indefinite character* for (5). It is determined when the numerator and denominator in the right-hand side of (5) vanish simultaneously. All the singular solutions to (5) which are not branches of the same analytic function pass through this point $(0,0)$ [17, Chap.9, §1].

Regular and singular solutions to (5) behave according to the Poincaré-Bendixson theorems [13], [2], [17, Chap.9, §1]. Namely, two integral curves of differential equation (5) intersect only at the singular point $(0,0)$. An integral curve of (5) can have multiple points only at $(0,0)$. Bendixson [2] considered real integral curves globally and stated that they have endpoints at knots and focuses and have an extension through a saddle. Under these assumptions, the Bendixson theorem [2] makes possible only three cases for equation (5) in a neighborhood of $(0,0)$: (a) an integral curve is closed, i.e., it is a cycle; (b) an integral curve is a spiral which tends to a cycle asymptotically; (c) an integral curve has the endpoint at $(0,0)$.

Recall the integrability case [8] of the Löwner differential equation (2) with the square root forcing $\lambda(t) = c\sqrt{t}$. After changing variables $t \to \tau^2$, the singular point $(0,0)$ in this case is a saddle according to the Poincaré classification [13] for linear differential equations. From the other side, another integrability case [8] with the square root forcing $\lambda(t) = c\sqrt{1-t}$, after changing variables $t \to 1-\tau^2$, leads to the focus at $(0,0)$.

Going back to equation (5) remark that its solutions are infinitely differentiable with respect to the real variable τ, see [4, Chap.1, §1], [17, Chap.9, §1]. Hence recurrent evaluations of Taylor coefficients can help to find singular solutions provided that a resulting series will have a positive convergence radius [16, Chap.3, §1]. Apply this method to equation (5). Let

$$g_s(0,\tau) = \sum_{n=1}^{\infty} a_n \tau^n \tag{6}$$

be a a formal power series for singular solutions to (5). Note that g_s is not necessarily unique. It depends on the path along which z approaches to 0, $z \notin K_\tau$. Substitute (6) into (5) and see that

$$\sum_{n=1}^{\infty} n a_n \tau^{n-1} \left(\sum_{n=1}^{\infty} a_n \tau^n - \tau \right) = 6\tau^2. \tag{7}$$

Equating coefficients at the same powers in both sides of (7) obtain that

$$a_1(a_1 - 1) = 0. \tag{8}$$

This equation gives two possible values $a_1 = 1$ and $a_1 = 0$ to two singular solutions $g^+(0,\tau)$ and $g^-(0,\tau)$. In both cases equation (7) implies recurrent formulas for coefficients a_n^+ and a_n^- of $g^+(0,\tau)$ and $g^-(0,\tau)$ respectively,

$$a_1^+ = 1, \;\; a_2^+ = 6, \;\; a_n^+ = -\sum_{k=2}^{n-1} k a_k^+ a_{n+1-k}^+, \;\; n \geq 3, \tag{9}$$

$$a_1^- = 0, \;\; a_2^- = -3, \;\; a_n^- = \frac{1}{n}\sum_{k=2}^{n-1} k a_k^- a_{n+1-k}^-, \;\; n \geq 3, \tag{10}$$

Show that the series $\sum_{n=1}^{\infty} a_n^+ \tau^n$ formally representing $g^+(0,\tau)$ diverges for all $\tau \neq 0$.

Lemma 1. *For $n \geq 2$, the inequalities*

$$6^{n-1}(n-1)! \leq |a_n^+| \leq 12^{n-1} n^{n-3} \tag{11}$$

hold.

Proof. For $n = 2$, the estimate (11) from below holds with the equality sign. Suppose that these estimates are true for $k = 2, \dots, n-1$ and substitute them in (7). For $n \geq 3$, we have

$$|a_n^+| = \sum_{k=2}^{n-1} k|a_k^+||a_{n+1-k}^+| \geq \sum_{k=2}^{n-1} k 6^{k-1}(k-1)! 6^{n-k}(n-k)! =$$

$$6^{n-1}\sum_{k=2}^{n-1} k!(n-k)! \geq 6^{n-1}(n-1)!\,.$$

This confirms by induction the estimate (11) from below.

Similarly, for $n = 2, 3$, the estimate (11) from above is easily verified. Suppose that these estimates are true for $k = 2, \dots, n-1$ and substitute them in (7). For $n \geq 4$, we have

$$|a_n^+| = \sum_{k=2}^{n-1} k|a_k^+||a_{n+1-k}^+| \leq \sum_{k=2}^{n-1} k 12^{k-1} k^{k-3} 12^{n-k}(n+1-k)^{n-2-k} =$$

$$12^{n-1}\sum_{k=2}^{n-1} k^{k-2}(n+1-k)^{n-2-k} \leq 12^{n-1}\left(\sum_{k=2}^{n-2}(n-1)^{k-2}(n-1)^{n-2-k} + \frac{(n-1)^{n-3}}{2}\right)$$

$$< 12^{n-1}\left(\sum_{k=2}^{n-2}(n-1)^{n-4} + (n-1)^{n-4}\right) < 12^{n-1} n^{n-3}$$

which completes the proof. $\square$

Evidently, the upper estimates (11) are preserved for $|a_n^-|$, $n \geq 2$.

The lower estimates (11) imply divergence of $\sum_{n=1}^{\infty} a_n^+ \tau^n$ for $\tau \neq 0$. Therefore equation (5) does not have any holomorphic solution in a neighborhood of $\tau_0 = 0$.

There exist some methods to summarize the series $\sum_{n=1}^{\infty} a_n^+ \tau^n$, the Borel regular method among them [3], [16, Chap.3, §1]. Let

$$G(\tau) = \sum_{n=1}^{\infty} \frac{a_n^+}{n!} \tau^n,$$

this series converges for $|\tau| < 1/12$ according to Lemma 1. The Borel sum equals

$$h(\tau) = \int_0^{\infty} e^{-x} G(\tau x) dx$$

and solves (5) provided it determines an analytic function. The same approach is applied to $\sum_{n=1}^{\infty} a_n^- \tau^n$.

In any case solutions $g_1(0,\tau)$, $g_2(0,\tau)$ to (5) emanating from the singular point $(0,0)$ satisfy the asymptotic relations

$$g_1(0,\tau) = \sum_{k=1}^{n} a_k^+ \tau^k + o(\tau^n), \quad g_2(0,\tau) = \sum_{k=1}^{n} a_k^- \tau^k + o(\tau^n), \quad \tau \to 0,$$

for all $n \geq 2$, $o(\tau^n)$ in both representations depend on n.

Let $f_1(0,t) := g_1(0,\tau^3)$, $f_2(0,t) := g_2(0,\tau^3)$. Since $f_1(0,t) = \sqrt[3]{t} + 6\sqrt[3]{t^2} + o(\sqrt[3]{t^2})$ and $f_2(0,t) = -3\sqrt[3]{t^2} + o(\sqrt[3]{t^2})$ as $t \to 0$, the inequality

$$f_2(0,t) < \sqrt[3]{t} < f_1(0,t)$$

holds for all $t > 0$ small enough.

Let us find representations for all other singular solutions to equation (4) which appear at $t > 0$. Suppose there is $z_0 \in \mathbb{H}$ and $t_0 > 0$ such that $f(z_0,t_0) = \sqrt[3]{t}$. Then $(f(z_0,t_0),t_0)$ is a singular point of equation (4), and $f(z_0,t)$ is expanded in series with powers $(t-t_0)^{n/m}$, $m \in \mathbb{N}$,

$$f(z_0,t) = \sqrt[3]{t_0} + \sum_{n=1}^{\infty} b_{n/m}(t-t_0)^{n/m}. \tag{12}$$

Substitute (12) into (4) and see that

$$\sum_{n=1}^{\infty} \frac{n b_{n/m}(t-t_0)^{n/m-1}}{m} \times$$

$$\left(\sum_{n=1}^{\infty} b_{n/m}(t-t_0)^{n/m} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2 \cdot 5 \ldots (3n-4)}{n!} \frac{(t-t_0)^n}{(3t_0)^n} \right) = 2. \tag{13}$$

Equating coefficients at the same powers in both sides of (13) obtain that $m = 2$ and

$$(b_{1/2})^2 = 4. \tag{14}$$

This equation gives two possible values $b_{1/2} = 2$ and $b_{1/2} = -2$ to two branches $f_1(z_0,t)$ and $f_2(z_0,t)$ of the solution (12). Indeed, we can accept only one of possibilities, for example $b_{1/2} = 2$, while the second case is obtained by going to another

branch of $(t-t_0)^{n/2}$ when passing through $t=t_0$. So we have recurrent formulas for coefficients $b_{n/2}$ of $f_1(z_0,t)$ and $f_2(z_0,t)$,

$$b_{1/2}=2,\quad b_{n/2}=\frac{1}{n+1}\left(c_{n/2}-\frac{1}{2}\sum_{k=2}^{n-1}kb_{k/2}(b_{(n+1-k)/2}-c_{(n+1-k)/2})\right),\quad n\geq 2, \tag{15}$$

where

$$c_{(2k-1)/2}=0,\quad c_k=\frac{(-1)^{k-1}2\cdot 5\dots(3k-4)}{3^k t_0^{k-1/3}k!},\quad k=1,2,\dots . \tag{16}$$

Since

$$f_1(z_0,t)=\sqrt[3]{t_0}+2\sqrt{t-t_0}+o(\sqrt{t-t_0}),\ f_2(z_0,t)=\sqrt[3]{t_0}-2\sqrt{t-t_0}+o(\sqrt{t-t_0}),$$

$$\sqrt[3]{t}=\sqrt[3]{t_0}+\frac{1}{3t_0}(t-t_0)+o(t-t_0),\quad t\to t_0+0,$$

the inequality

$$f_2(z_0,t)<\sqrt[3]{t}<f_1(z_0,t)$$

holds for all $t>t_0$ close to t_0.

3. Proof of the main results

The theory of differential equations claims that integral curves of equation (4) intersect only at the singular point $(0,0)$ [17, Chap.9, §1]. In particular, this implies the local inequalities $f_2(0,t)<f_2(z_0,t)<\sqrt[3]{t}<f_1(z_0,t)<f_1(0,t)$ where $(f(z_0,t_0),t_0)$ is an algebraic solution critical point for equation (4). We will give an independent proof of these inequalities which can be useful for more general driving functions.

Lemma 2. *For $t>0$ small enough and a singular point $(f(z_0,t_0),t_0)$ for equation (4), $0<t_0<t$, the following inequalities*

$$f_2(0,t)<f_2(z_0,t)<\sqrt[3]{t}<f_1(z_0,t)<f_1(0,t)$$

hold.

Proof. To show that $f_1(z_0,t)<f_1(0,t)$ let us subtract equations

$$\frac{df_1(0,t)}{dt}=\frac{2}{f_1(0,t)-\sqrt[3]{t}},\quad f_1(0,0)=0,$$

$$\frac{df_1(z_0,t)}{dt}=\frac{2}{f_1(z_0,t)-\sqrt[3]{t}},\quad f_1(z_0,t_0)=\sqrt[3]{t_0},$$

and obtain

$$\frac{d(f_1(0,t)-f_1(z_0,t))}{dt}=\frac{2(f_1(z_0,t)-f_1(0,t))}{(f_1(0,t)-\sqrt[3]{t})(f_1(z_0,t)-\sqrt[3]{t})},$$

which can be written in the form

$$\frac{d\log(f_1(0,t)-f_1(z_0,t))}{dt}=\frac{-2}{(f_1(0,t)-\sqrt[3]{t})(f_1(z_0,t)-\sqrt[3]{t})}.$$

Suppose that $T > t_0$ is the smallest number for which $f_1(0,T) = f_1(z_0,T)$. This implies that

$$\int_{t_0}^{T} \frac{dt}{(f_1(0,t) - \sqrt[3]{t})(f_1(z_0,t) - \sqrt[3]{t})} = \infty. \tag{17}$$

To evaluate the integral in (17) we should study the behavior of $f_1(z_0,t) - \sqrt[3]{t}$ with the help of differential equation

$$\frac{d(f_1(z_0,t) - \sqrt[3]{t})}{dt} = \frac{2}{f_1(z_0,t) - \sqrt[3]{t}} - \frac{1}{3\sqrt[3]{t^2}} = \frac{\sqrt[3]{t} + 6\sqrt[3]{t^2} - f_1(z_0,t)}{3\sqrt[3]{t^2}(f_1(z_0,t) - \sqrt[3]{t})}. \tag{18}$$

Calculate that $a_3^+ = -72$ and write the asymptotic relation

$$f_1(0,t) = \sqrt[3]{t} + 6\sqrt[3]{t^2} - 72t + o(t), \quad t \to +0.$$

There exists a number $T' > 0$ such that for $0 < t < T'$, $\sqrt[3]{t} + 6\sqrt[3]{t^2} > f_1(0,t)$. Consequently, the right-hand side in (18) is positive for $0 < t < T'$. Note that T' does not depend on t_0. The condition "$t > 0$ small enough" in Lemma 2 is understood from now as $0 < t < T'$. We see from (18) that for such t, $f_1(z_0,t) - \sqrt[3]{t}$ is increasing with t, $t_0 < t < T < T'$. Therefore, the integral in the left-hand side of (17) is finite. The contradiction against equality (17) denies the existence of T with the prescribed properties which proves the third and the fourth inequalities in Lemma 2.

The rest of inequalities in Lemma 2 are proved similarly and even easier. To show that $f_2(z_0,t) > f_2(0,t)$ let us subtract equations

$$\frac{df_2(0,t)}{dt} = \frac{2}{f_2(0,t) - \sqrt[3]{t}}, \quad f_2(0,0) = 0,$$

$$\frac{df_2(z_0,t)}{dt} = \frac{2}{f_2(z_0,t) - \sqrt[3]{t}}, \quad f_2(z_0,t_0) = \sqrt[3]{t_0},$$

and obtain

$$\frac{d(f_2(0,t) - f_2(z_0,t))}{dt} = \frac{2(f_2(z_0,t) - f_2(0,t))}{(f_2(0,t) - \sqrt[3]{t})(f_2(z_0,t) - \sqrt[3]{t})},$$

which can be written in the form

$$\frac{d\log(f_2(z_0,t) - f_2(0,t))}{dt} = \frac{-2}{(f_2(0,t) - \sqrt[3]{t})(f_2(z_0,t) - \sqrt[3]{t})}.$$

Suppose that $T > t_0$ is the smallest number for which $f_2(z_0,T) = f_2(0,T)$. This implies that

$$\int_{t_0}^{T} \frac{dt}{(f_2(0,t) - \sqrt[3]{t})(f_2(z_0,t) - \sqrt[3]{t})} = \infty. \tag{19}$$

To evaluate the integral in (19) we should study the behavior of $f_2(z_0,t)-\sqrt[3]{t}$ with the help of differential equation

$$\frac{d(f_2(z_0,t)-\sqrt[3]{t})}{dt}=\frac{2}{f_2(z_0,t)-\sqrt[3]{t}}-\frac{1}{3\sqrt[3]{t^2}}=\frac{\sqrt[3]{t}+6\sqrt[3]{t^2}-f_2(z_0,t)}{3\sqrt[3]{t^2}(f_2(z_0,t)-\sqrt[3]{t})}. \tag{20}$$

Since

$$f_2(0,t)=-3\sqrt[3]{t^2}+o(\sqrt[3]{t^2}),\quad t\to+0,$$

there exists a number $T''>0$ such that for $0<t<T''$, $\sqrt[3]{t}+6\sqrt[3]{t^2}>f_2(0,t)$. Consequently, the right-hand side in (20) is positive for $0<t<T''$. We see from (20) that for such t, $f_2(0,t)-\sqrt[3]{t}$ is decreasing with t, $t_0<t<T<T''$. Therefore, the integral in the left-hand side of (19) is finite. The contradiction against equality (19) denies the existence of T with the prescribed properties which completes the proof. □

Add and complete the inequalities of Lemma 2 by the following statements demonstrating a monotonic disposition of values for different singular solutions.

Lemma 3. *For $t>0$ small enough and singular points $(f(z_1,t_1),t_1)$, $(f(z_0,t_0),t_0)$ for equation (4), $0<t_1<t_0<t$, the following inequalities*

$$f_2(z_1,t)<f_2(z_0,t),\quad f_1(z_0,t)<f_1(z_1,t)$$

hold.

Proof. Similarly to Lemma 2, subtract equations

$$\frac{df_1(z_1,t)}{dt}=\frac{2}{f_1(z_1,t)-\sqrt[3]{t}},\quad f_1(z_1,t_1)=\sqrt[3]{t_1},$$

$$\frac{df_1(z_0,t)}{dt}=\frac{2}{f_1(z_0,t)-\sqrt[3]{t}},\quad f_1(z_0,t_0)=\sqrt[3]{t_0},$$

and obtain

$$\frac{d(f_1(z_1,t)-f_1(z_0,t))}{dt}=\frac{2(f_1(z_0,t)-f_1(z_1,t))}{(f_1(z_1,t)-\sqrt[3]{t})(f_1(z_0,t)-\sqrt[3]{t})},$$

which can be written in the form

$$\frac{d\log(f_1(z_1,t)-f_1(z_0,t))}{dt}=\frac{-2}{(f_1(z_1,t)-\sqrt[3]{t})(f_1(z_0,t)-\sqrt[3]{t})}.$$

Suppose that $T>t_0$ is the smallest number for which $f_1(z_1,T)=f_1(z_0,T)$. This implies that

$$\int_{t_0}^{T}\frac{dt}{(f_1(z_1,t)-\sqrt[3]{t})(f_1(z_0,t)-\sqrt[3]{t})}=\infty. \tag{21}$$

To evaluate the integral in (21) apply to (18) and obtain that there exists a number $T'>0$ such that for $0<t<T'$, $f_1(z_0,t)-\sqrt[3]{t}$ is increasing with t, $t_0<t<T<T'$. Therefore, the integral in the left-hand side of (21) is finite. The contradiction

against equality (21) denies the existence of T with the prescribed properties which proves the second inequality of Lemma 3.

To prove the first inequality of Lemma 3 subtract equations

$$\frac{df_2(z_1,t)}{dt} = \frac{2}{f_2(z_1,t)-\sqrt[3]{t}}, \quad f_2(z_1,t_1) = \sqrt[3]{t_1},$$
$$\frac{df_2(z_0,t)}{dt} = \frac{2}{f_2(z_0,t)-\sqrt[3]{t}}, \quad f_2(z_0,t_0) = \sqrt[3]{t_0},$$

and obtain after dividing by $f_2(z_1,t) - f_2(z_0,t)$

$$\frac{d\log(f_2(z_0,t)-f_2(z_1,t))}{dt} = \frac{-2}{(f_2(z_1,t)-\sqrt[3]{t})(f_2(z_0,t)-\sqrt[3]{t})}.$$

Suppose that $T > t_0$ is the smallest number for which $f_2(z_0,T) = f_2(z_1,T)$. This implies that

$$\int_{t_0}^{T} \frac{dt}{(f_2(z_1,t)-\sqrt[3]{t})(f_2(z_0,t)-\sqrt[3]{t})} = \infty. \tag{22}$$

To evaluate the integral in (22) apply to (20) and obtain that $\sqrt[3]{t} + 6\sqrt[3]{t^2} > \sqrt[3]{t} > f_2(0,t)$. Consequently, the right-hand side in (20) is positive and we see that $f_2(0,t) - \sqrt[3]{t}$ is decreasing with t, $t_0 < t < T$. Therefore, the integral in the left-hand side of (22) is finite. The contradiction against equality (22) denies the existence of T with the prescribed properties which completes the proof. □

Proof of Theorem 1.

For $t_0 > 0$, there is a hull $K_{t_0} \subset \mathbb{H}$ such that $f(\cdot,t_0)$ maps $\mathbb{H}\setminus K_{t_0}$ onto $\mathbb{H}$. We refer to [10] for definitions and more details. The hull K_{t_0} is driven by $\sqrt[3]{t}$. The function $f(\cdot,t_0)$ is extended continuously onto the set of prime ends on $\partial(\mathbb{H}\setminus K_{t_0})$ and maps this set onto $\mathbb{R}$. One of the prime ends is mapped on $\sqrt[3]{t_0}$. Let $z_0 = z_0(t_0)$ represent this prime end.

Lemmas 2 and 3 describe the structure of the pre-image of $\mathbb{H}$ under $f(\cdot,t)$. All the singular solutions $f_1(0,t)$, $f_2(0,t)$, $f_1(z_0,t)$, $f_2(z_0,t)$, $0 < t_0 < t < T'$, are real-valued and satisfy the inequalities of Lemmas 2 and 3. So the segment $I = [f_2(0,t), f_1(0,t)]$ is the union of the segments $I_2 = [f_2(0,t), \sqrt[3]{t}]$ and $I_1 = [\sqrt[3]{t}, f_1(0,t)]$. The segment I_2 consists of points $f_2(z(\tau),t)$, $0 \le \tau < t$, and the segment I_1 consists of points $f_1(z(\tau),t)$, $0 \le \tau < t$. All these points belong to the boundary $\mathbb{R} = \partial\mathbb{H}$. This means that all the points $z(\tau)$, $0 \le \tau < t$, belong to the boundary $\partial(\mathbb{H}\setminus K_t)$ of $\mathbb{H}\setminus K_t$. Moreover, every point $z(\tau)$ except for the tip determines exactly two prime ends corresponding to $f_1(z(\tau),t)$ and $f_2(z(\tau),t)$. Evidently, $z(\tau)$ is continuous on $[0,t]$. This proves that $z(\tau) := \gamma(\tau)$ represents a curve $\gamma := K_t$ with prime ends corresponding to points on different sides of γ. This proves that $f^{-1}(w,t)$ maps $\mathbb{H}$ onto the slit domain $\mathbb{H}\setminus\gamma(t)$ for $t > 0$ small enough.

It remains to show that $\gamma(t)$ is a C^1-curve. Fix $t_0 > 0$ from a neighborhood of $t = 0$. Denote $g(w,t) = f^{-1}(w,t)$ an inverse of $f(z,t)$, and $h(w,t) := f(g(w,t_0),t)$, $t \ge t_0$. The arc $\gamma[t_0,t] := K_t \setminus K_{t_0}$ is mapped by $f(z,t_0)$ onto a curve $\gamma_1(t)$ in $\mathbb{H}$

emanating from $\sqrt[3]{t_0} \in \mathbb{R}$. So the function $h(w,t)$ is well defined on $\mathbb{H}\setminus\gamma_1(t_0)$, $t \geq t_0$. Expand $h(w,t)$ near infinity,

$$h(w,t) = g(w,t_0) + \frac{2t}{g(w,t_0)} + O\left(\frac{1}{g^2(w,t_0)}\right) = w + \frac{2(t-t_0)}{w} + O\left(\frac{1}{w^2}\right).$$

Such expansion satisfies (1) after changing variables $t \to t - t_0$. The function $h(w,t)$ satisfies the differential equation

$$\frac{dh(w,t)}{dt} = \frac{2}{h(w,t) - \sqrt[3]{t}}, \quad h(w,t_0) = w, \quad w \in \mathbb{H}.$$

This equation becomes the Löwner differential equation if $t_1 := t - t_0$, $h_1(w,t_1) := h(w, t_0 + t_1)$,

$$\frac{dh_1(w,t_1)}{dt_1} = \frac{2}{h_1(w,t_1) - \sqrt[3]{t_1 + t_0}}, \quad h_1(w,0) = w, \quad w \in \mathbb{H}. \tag{23}$$

The driving function $\lambda(t_1) = \sqrt[3]{t_1 + t_0}$ in (23) is analytic for $t_1 \geq 0$. It is known [1, p.59] that under this condition $h_1(w,t_1)$ maps $\mathbb{H}\setminus\gamma_1$ onto $\mathbb{H}$ where γ_1 is a C^1-curve in $\mathbb{H}$ emanating from $\lambda(0) = \sqrt[3]{t_0}$. The same does the function $h(w,t)$.

Go back to $f(z,t) = h(f(z,t_0),t)$ and see that $f(z,t)$ maps $\mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$, $\gamma(t) = \gamma[0,t_0] \cup \gamma[t_0,t]$, and $\gamma[t_0,t]$ is a C^1-curve. Tending t_0 to 0 we prove that $\gamma(t)$ is a C^1-curve, except probably for the point $\gamma(0) = 0$. This completes the proof.

4. Harmonic measures of the slit sides

The function $f(z,t)$ solving (4) maps $\mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$. The curve $\gamma(t)$ has two sides. Denote $\gamma_1 = \gamma_1(t)$ the side of γ which is mapped by the extended function $f(z,t)$ onto $I_1 = [\sqrt[3]{t}, f_1(0,t)]$. Similarly, $\gamma_2 = \gamma_2(t)$ is the side of γ which is the pre-image of $I_2 = [f_2(0,t), \sqrt[3]{t}]$ under $f(z,t)$.

Remind that the harmonic measures $\omega(f^{-1}(i,t); \gamma_k, \mathbb{H} \setminus \gamma(t), t)$ of γ_k at $f^{-1}(i,t)$ with respect to $\mathbb{H} \setminus \gamma(t)$ are defined by the functions ω_k which are harmonic on $\mathbb{H} \setminus \gamma(t)$ and continuously extended on its closure except for the endpoints of γ, $\omega_k|_{\gamma_k(t)} = 1$, $\omega_k|_{\mathbb{R}\cup(\gamma(t)\setminus\gamma_k(t))} = 0$, $k = 1,2$, see, e.g., [6, Chap.3, §3.6]. Denote

$$m_k(t) := \omega(f^{-1}(i,t); \gamma_k, \mathbb{H} \setminus \gamma(t), t), \quad k = 1,2.$$

Theorem 2. *Let $f(z,t)$ be a solution to the Löwner equation (4). Then*

$$\lim_{t\to+0} \frac{m_1(t)}{m_2^2(t)} = 6\pi. \tag{24}$$

Proof. The harmonic measure is invariant under conformal transformations. So

$$\omega(f^{-1}(i,t); \gamma_k, \mathbb{H} \setminus \gamma(t), t) = \Omega(i; f(\gamma_k,t), \mathbb{H}, t)$$

are defined by the harmonic functions Ω_k which are harmonic on $\mathbb{H}$ and continuously extended on $\mathbb{R}$ except for the endpoints of $f(\gamma_k,t)$, $\Omega_k|_{f(\gamma_k,t)} = 1$, $\Omega_k|_{\mathbb{R}\setminus f(\gamma_k,t)} = 0$, $k = 1,2$. The solution of the problem is known, see, e.g., [5, p.334]. Namely,

$$m_k(t) = \frac{\alpha_k(t)}{\pi}$$

where $\alpha_k(t)$ is the angle under which the segment $I_k = I_k(t)$ is observed from the point $w = i$, $k = 1, 2$. It remains to find asymptotic expansions for $\alpha_k(t)$.

Since

$$f_1(0,t) = \sqrt[3]{t} + 6\sqrt[3]{t^2} + O(t), \quad f_2(0,t) = -3\sqrt[3]{t^2} + O(t), \quad t \to +0,$$

after elementary geometrical considerations we have

$$\alpha_1(t) = \arctan f_1(0,t) - \arctan \sqrt[3]{t} = 6\sqrt[3]{t^2} + O(t), \quad t \to +0,$$

$$\alpha_2(t) = \arctan \sqrt[3]{t} - \arctan f_2(0,t) = \sqrt[3]{t} + 3\sqrt[3]{t^2} + O(t), \quad t \to +0.$$

This implies that

$$\frac{m_1(t)}{m_2^2(t)} = \pi \frac{6\sqrt[3]{t^2} + O(t)}{(\sqrt[3]{t} + 3\sqrt[3]{t^2} + O(t))^2} = 6\pi(1 + O(\sqrt[3]{t})), \quad t \to +0,$$

which leads to (24) and completes the proof. □

Remark 1. *The relation similar to (24) follows from* [14] *for the two sides of the circular slit $\gamma(t)$ in $\mathbb{H}$ such that $\gamma(t)$ is tangential to $\mathbb{R}$ at $z = 0$.*

5. Representation of holomorphic solutions

Holomorphic solutions to (4) or, equivalently, to (5) appear in a neighborhood of every non-singular point $(z_0, 0)$. We will be interested in real solutions corresponding to $z_0 \in \mathbb{R}$.

Put $z_0 = \epsilon > 0$ and let

$$f(\epsilon, t) = \epsilon + \sum_{n=1}^{\infty} a_n(\epsilon) t^{n/3} \tag{25}$$

be a solution of equation (4) holomorphic with respect to $\tau = \sqrt[3]{t}$. Change $\sqrt[3]{t}$ by τ and substitute (25) in (5) to get that

$$\sum_{n=1}^{\infty} n a_n(\epsilon) \tau^{n-1} \left[\epsilon - \tau + \sum_{n=1}^{\infty} a_n(\epsilon) \tau^n \right] = 6\tau^2. \tag{26}$$

Equate coefficients at the same powers in both sides of (26) and obtain equations

$$a_1(\epsilon) = 0, \quad a_2(\epsilon) = 0, \quad a_k(\epsilon) = \frac{6}{k\epsilon^{k-2}}, \quad k = 3, 4, 5, \tag{27}$$

and

$$a_n(\epsilon) = \frac{1}{n\epsilon} \left[(n-1) a_{n-1}(\epsilon) - \sum_{k=3}^{n-3} (n-k) a_{n-k}(\epsilon) a_k(\epsilon) \right], \quad n \geq 6. \tag{28}$$

The series in (25) converges for $|\tau| = |\sqrt[3]{t}| < R(\epsilon)$.

Theorem 3. *The series in (25) converges for*

$$|t| < \epsilon^3 + o(\epsilon^3), \quad \epsilon \to +0. \tag{29}$$

Proof. Estimate the convergence radius $R(\epsilon)$ following the Cauchy majorant method, see, e.g., [4, Chap.1, §§2-3], [16, Chap.3, §1]. The Cauchy theorem states: if the right-hand side in (5) is holomorphic on a product of the closed disks $|g-\epsilon|\le\rho_1$ and $|\tau|\le r_1$ and is bounded there by M, then the series $\sum_{n=1}^{\infty}a_n(\epsilon)\tau^n$ converges in the disk

$$|\tau|<R(\epsilon)=r_1\left(1-\exp\left\{-\frac{\rho_1}{2Mr_1}\right\}\right).$$

In the case of equation (5) we have

$$\rho_1+r_1<\epsilon,\quad\text{and}\quad M=\frac{6r_1^2}{\epsilon-(\rho_1+r_1)}.$$

This implies that for $\rho_1+r_1=\epsilon-\delta$, $\delta>0$,

$$R(\epsilon)=r_1\left(1-\exp\left\{-\frac{\epsilon-\delta-r_1}{12r_1^2}\delta\right\}\right).$$

So $R(\epsilon)$ depends on δ and r_1. Maximum of R with respect to δ is obtained for $\delta=(\epsilon-r_1)/2$. Hence, this maximum is equal to

$$R_1(\epsilon)=r_1\left(1-\exp\left\{-\frac{(\epsilon-r_1)^2}{48r_1^3}\right\}\right),\tag{30}$$

where $R_1(\epsilon)$ depends on r_1. Let us find a maximum of R_1 with respect to $r_1\in(0,\epsilon)$. Notice that R_1 vanishes for $r_1=0$ and $r_1=\epsilon$. Therefore the maximum of R_1 is attained for a certain root $r_1=r_1(\epsilon)\in(0,\epsilon)$ of the derivative of R_1 with respect to r_1. To simplify the calculations we put $r_1(\epsilon)=\epsilon c(\epsilon)$, $0<c(\epsilon)<1$. Now the derivative of R_1 vanishes for $c=c(\epsilon)$ satisfying

$$1-\exp\left\{-\frac{(1-c)^2}{48\epsilon c^3}\right\}\left(1+\frac{(1-c)(3-c)}{48\epsilon c^3}\right)=0.\tag{31}$$

Choose a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty}\epsilon_n=0$, such that $c(\epsilon_n)$ converge to c_0 as $n\to\infty$. Suppose that $c_0<1$. Then

$$\exp\left\{-\frac{(1-c(\epsilon_n))^2}{48\epsilon_n c^3(\epsilon_n)}\right\}\left(1+\frac{(1-c(\epsilon_n))(3-c(\epsilon_n))}{48\epsilon_n c^3(\epsilon_n)}\right)<1$$

for n large enough. Therefore $c(\epsilon_n)$ is not a root of equation (31) for $\epsilon=\epsilon_n$ and n large enough. This contradiction claims that $c_0=1$ for every sequence $\{\epsilon_n>0\}_{n=1}^{\infty}$ tending to 0 with $\lim_{n\to\infty}c(\epsilon_n)=c_0$. So we proved that $c(\epsilon)\to1$ as $\epsilon\to+0$.

Consequently, the maximum of R_1 with respect to r_1 is attained for $r_1(\epsilon)=\epsilon c(\epsilon)=\epsilon(1+o(1))$ as $\epsilon\to+0$. Let $R_2=R_2(\epsilon)$ denote the maximum of R_1 with respect to r_1. It follows from (30) that

$$R_2(\epsilon)=r_1(\epsilon)\left(1-\exp\left\{-\frac{(\epsilon-r_1(\epsilon))^2}{48r_1^3(\epsilon)}\right\}\right)=\epsilon c(\epsilon)\left(1-\exp\left\{-\frac{(1-c(\epsilon))^2}{48\epsilon c^3(\epsilon)}\right\}\right).\tag{32}$$

Examine how fast does $c(\epsilon)$ tends to 1 as $\epsilon\to+0$. Choose a sequence $\{\epsilon_n>0\}_{n=1}^{\infty}$, $\lim_{n\to\infty}\epsilon_n=0$, such that the sequence $(1-c(\epsilon_n))^2/\epsilon_n$ converges to a non-negative

number or to ∞. Denote

$$l := \lim_{n\to\infty} \frac{(1-c(\epsilon_n))^2}{\epsilon_n}, \quad 0 \le l \le \infty.$$

If $0 < l < \infty$, then $(1-c(\epsilon_n))/\epsilon_n$ tends to ∞, and equation (31) with $\epsilon = \epsilon_n$ has no roots for n large enough.

If $l = 0$, then, according to (31), $\lim_{n\to\infty}(1-c(\epsilon_n))/\epsilon_n = 0$, and

$$\exp\left\{-\frac{(1-c(\epsilon_n))^2}{48\epsilon c^3(\epsilon)}\right\}\left(1+\frac{(1-c(\epsilon_n))(3-c(\epsilon_n))}{48\epsilon_n c^3(\epsilon_n)}\right) =$$

$$\left(1-\frac{(1-c(\epsilon_n))^2}{48\epsilon_n c^3(\epsilon_n)} + o\left(\frac{(1-c(\epsilon_n))^2}{\epsilon_n}\right)\right)\left(1+\frac{(1-c(\epsilon_n))(3-c(\epsilon_n))}{48\epsilon_n c^3(\epsilon_n)} +\right) =$$

$$1+\frac{1-c(\epsilon_n)}{24\epsilon_n} + o\left(\frac{1-c(\epsilon_n)}{\epsilon_n}\right), \quad n\to\infty.$$

This implies again that equation (31) with $\epsilon = \epsilon_n$ has no roots for n large enough.

Thus the only possible case is $l = \infty$ for all sequences $\{\epsilon_n > 0\}_{n=1}^{\infty}$ converging to 0. It follows from (32) that

$$R_2(\epsilon) = \max_{0<r_1(\epsilon)<\epsilon} R_1(\epsilon) = \epsilon + o(\epsilon), \quad \epsilon \to 0. \tag{33}$$

In other words, the series in (25) converges for $|t| < (\epsilon + o(\epsilon))^3$, $\epsilon \to 0$, which implies the statement of Theorem 3 and completes the proof. $\square$

Remark 2. *Evidently, a similar conclusion with the same formulas (27) and (28) is true for $\epsilon < 0$.*

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